A. Khovanskii, A. Tronsgard, On solvability of linear differential equations in finite terms,
Изв. РАН. Сер. матем., 2026, том 90, выпуск 1, 244–260

https://www.mathnet.ru/eng/im9686

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On solvability of linear differential equations in finite terms

We consider the problem of solvability of linear differential equations over a differential field K. We introduce a class of special differential field extensions, which widely generalizes the classical class of extensions of differential fields by integrals and by exponentials of integrals and which has similar properties. We announce the following result: if a linear differential equation over K cannot be solved by generalized quadratures, then no special extension can help solve it. In the paper, we prove a weaker version of this result in which we consider only pure transcendental extensions of K. Our paper is self-contained and understandable for beginners. It demonstrates the power of Liouville's original approach to problems of solvability of equations in finite terms.

Bibliography: 8 titles.

Keywords: linear differential equations, solvability by quadratures, integration in finite terms, differential Galois theory.

DOI: https://doi.org/10.4213/im9686

To the memory of Andrei Bolibrukh

§ 1. Introduction

In the 1830s, Liouville started to create a theory of solvability in finite terms. In some cases it answers the following question: is there a solution of an equation which is representable by a certain class of formulas? In other words, is there a solution representable in *finite terms*?

About the same time Galois theory was invented. It provides a criterion of solvability of algebraic equations in radicals. Liouville was inspired by Galois theory. Among his results there are two famous theorems. The First Liouville Theorem provides a criterion of integrability of elementary functions in elementary functions. The Second Liouville Theorem provides a criterion of solvability of second order linear differential equations by quadratures. Modern proofs of these theorems can be found respectively in [1] and [2].

Liouville's brilliant and simple ideas were never properly understood (maybe, because Liouville did not carefully formalize them). Only in 1910, Picard and Vessiot generalized the Second Liouville Theorem for linear differential equations

The first author was partially supported by the Canadian Grant No. 156833-17. The second author was partially supported by the NSERC post-graduate scholarship (PGS-D) 569475.

AMS 2020 *Mathematics Subject Classification*. 12H20.

of an arbitrary order. To obtain this result they developed a new deep differential Galois theory.

J. F. Ritt up to some extent formalized Liouville's method and generalized it (see [3] and [4]). Later J. F. Ritt, E. R. Kolchin and others developed differential algebra, which totally gets rid of multi-valued analytic functions and deals with abstract differential fields. It became a well established branch of pure algebra.

If one wants to show that a solution of an equation is not representable by any formula of a certain type (for example, is not representable by quadratures), one has to deal with the collection of all functions representable by such formulas. Functions from this collection could be very complicated; they could be multi-valued, they could have complicated singularities, and so on. Such functions do not form a differential field, since arithmetic operations are not well defined for multi-valued functions.

In the framework of differential algebra, one can state and solve the problem of solvability of linear differential equations by quadratures. These results are applicable to analytic theory of differential equations in the following way.

Let K be a subfield of the field M of meromorphic functions on a connected open set U on a complex line, which is closed under differentiation. Consider a homogeneous linear differential equation over the field K

$$y^{(n)} + a_1 y^{n-1} + \cdots + a_n y = 0, \tag{1}$$

where $a_i \in K$. With a point $z \in U$ which is a regular point for all coefficients a_i, one can associate: the differential field K_z of germs at z of functions from K, the differential field F_z generated over K_z by germs at z of all solutions of (1), and the differential field M_z of all meromorphic germs at z.

Definition 1.1. An equation (1) is solvable by quadratures over K if there exists a point $z \in U$, such that inside the field M_z there is a chain of extensions by quadratures of the field K_z which contains the field F_z.

One can show that the definition of solvability of an equation by quadratures which makes use of multi-valued analytic functions is equivalent to Definition 1.1 (see [2], [5]).

Thus, results from differential algebra are applicable to the analytic theory of differential equations. Moreover, in differential algebra one can consider equations over arbitrary differential fields, not only over a field of meromorphic functions.

On the other hand, there are operations over analytic functions which do not exist in differential algebra. For example, one can compose functions, and it is natural to use compositions in formulas. Problems on solvability of equations by formulas involving compositions cannot be stated and solved using a pure algebraic approach. Note that compositions with some functions can be defined in differential algebra (see Remark 2.1).

A reduction of the Second Liouville Theorem (and its generalizations for linear differential equations of arbitrary order) to differential algebra should, in principal, significantly simplify it: it means that the result can be proved using only arithmetic operations and differentiation. Nevertheless, algebraic results on solvability of linear differential equations of arbitrary order, which uses differential Galois theory (see [6]) and Rosenlicht's algebraic proof (see [7]) of that result, which uses valuation theory, are rather involved. In [2], the same result over functional differential fields is proved using Liouville's original method. This proof is relatively simple, but it uses analytic tools and is not applicable to abstract differential fields.

Our goal is to find an algebraic proof of that result which is as simple as the original proof of the Second Liouville's Theorem. We are also looking for possible generalizations of that result.

We have achieved our goal. The result is announced in the next section. Currently, one step in our proof which deals with algebraic extensions is not polished and is too involved. We have decided in this first publication, to exclude algebraic extensions from consideration.

This makes our result weaker than the classical criterion of solvability of linear differential equations by quadratures. But also our result in many ways is stronger than the classical criterion: we show that many extensions (which we call special transcendental extensions) cannot help to solve linear differential equations. Besides that, our result is understandable for beginners and it clearly demonstrates Liouville's original approach.

In subsequent publications we plan to present a proof of the announced theorem. It is not as elementary as the present paper. We need the theory of algebraic curves over (possibly very big) algebraically closed fields. We also plan to consider much more general special extensions which have similar properties with special transcendental extensions.

§ 2. Solvability over differential fields

We begin with the definition of an abstract differential field.

Definition 2.1. A differential field is a field K along with a fixed additive map $K \to K$ sending $a \mapsto a'$ satisfying the Leibniz rule $(ab)' = a'b + ab'$. The element a' is called the derivative of $a \in K$.

Let us present several examples of differential fields.

Example 2.1. 1. The field $M(U)$ of all meromorphic functions on a connected open set U on the extended complex line $\mathbb{C}^1 \cup \{\infty\} = \mathbb{C}P^1$ with the natural differentiation.

2. The field of meromorphic functions $M(S)$ on a connected Riemann surface S equipped with non-constant meromorphic projection $\pi \colon S \to \mathbb{C}P^1$ with the following natural differentiation: for $f \in M(S)$ the derivative f' is defined as the ratio $df/d\pi$ of meromorphic differential forms df and $d\pi$.

3. Any subfield of the fields $M(U)$ and $M(S)$ closed under differentiation.

The differential field $\mathbb{C}(z)$ of rational functions of complex variable z and the differential field $\mathbb{C}(S)$ of rational functions on an algebraic curve S with non-constant rational projection $\pi \colon S \to \mathbb{C}P^1$ belong to these examples.

An element c of a differential field K is a *constant* if $c' = 0$. The collection of all constants in K form the *field of constants*.

Throughout the paper, by a differential field we mean a differential field of characteristic 0 with the same fixed subfield of constants. For example, one can assume that all differential fields under consideration contain $\mathbb{C}$ as the subfield of constants.

Consider a nested pair $K \subset F$ of differential fields.

DEFINITION 2.2. An element $y \in F$ is called

1) *algebraic over* K if y is a root of a polynomial over K;

2) an *integral over* K (or *an integral of* $a \in K$) if $y' = a$ and $a \in K$;

3) an *exponential of an integral over* K (or *an exponential of integral of* $a \in K$) if $y' = ay$ and $a \in K$.

REMARK 2.1. Many other natural definitions can be reformulated in such a way that they will make sense over differential fields. For example, the functions $u = \exp f$ and $v = \ln f$ satisfy the following relations: $u' = f'u$, $v' = f'/f$. Thus, one can say that the element u of F is an *exponential of* $f \in K$ if $u' = f'u$, and the element v of F is a *logarithm* of f if $v' = f'/f$.

Correspondingly, many different types of representability of functions in finite terms can be reformulated for differential fields.

Let us define a notion of solvability by quadratures in differential algebra. Consider a nested pair $K \subset F$ of differential fields. Let E be an intermediate differential field, i.e., $K \subset E \subset F$.

DEFINITION 2.3. An extension $K \subset E$ is an *extension by quadratures* if there is a chain of field extensions

$$K = K_0 \subset \cdots \subset K_n = E \tag{2}$$

such that, for each $0 \leqslant i < n$, the extension $K_i \subset K_{i+1}$ is either an extension by adjoining an integral over K_i, or by adjoining an exponential of an integral over K_i.

1. An element $a \in F$ is *representable by quadratures over* K if there is an extension $K \subset E$ by quadratures such that $a \in E$.

2. An equation over K is *solvable by quadratures over* K if there is an extension $K \subset E$ by quadratures which contains all solutions of the equation.

In the same way, one defines other types of solvability of equations in differential algebra.

DEFINITION 2.4. An extension $K \subset E$ is an *extension by generalized quadratures* if there is a chain (2) in which besides adjoining integrals and exponential of integrals, algebraic extensions are allowed.

1. An element $a \in F$ is *representable by generalized quadratures over* K if there is an extension $K \subset E$ by generalized quadratures such that $a \in E$.

2. An equation over K is *solvable by generalized quadratures over* K if there is an extension $K \subset E$ by generalized quadratures which contains all solutions of the equation.

Each extension $K_i \subset K_{i+1}$ in a chain (2) defining an extension by quadratures, or by generalized quadratures is generated over K_i by one element y_i. The element y_i is either algebraic, or transcendental over K_i. If y_i is algebraic, one can consider the extension $K_i \subset K_{i+1}$ as an algebraic equation. Otherwise, y_i has to satisfy an equation $y_i' = \mathcal{R}(y_i)$, where $\mathcal{R}$ is a rational function over K_i (see Lemma 6.3).

Definition 2.5. An element y is a *special transcendental element over* K if y is a transcendental element over K and it satisfies a differential equation $y' = \mathcal{R}(y)$, where $\mathcal{R}$ is a rational function over K which is divisible by y, i.e., is representable in the form $\mathcal{R} = yP/Q$, where the polynomials yP and Q are relatively prime.

Lemma 2.1. *If an element* y *is transcendental over* K *and either*

1) y *is an exponential of integral over* K,

2) y *is representable as* $y = u^{-1}$, *where* u *is an integral over* K,

then y *is a special transcendental element over* K.

Proof. 1) If $y' = ay$, then $y' = \mathcal{R}(y)$, where the rational function $\mathcal{R}(y) = ay$ is divisible by y.

2) If $u' = a$, then $y' = -ay^2$, i.e., $y' = \mathcal{R}(y)$, where the rational function $\mathcal{R}(y) = -ay^2$ is divisible by y.

Lemma 2.1 is proved.

The extension of K by u is identical to its extension by u^{-1}.

Definition 2.6. An extension $K \subset E$ is *admissible* if there is a chain (2) such that each extension $K_i \subset K_{i+1}$ is obtained by adjoining to K_i a special transcendental element over K_i.

Definition 2.7. An extension $K \subset E$ is a *generalized admissible extension* if there is a chain (2) such that each extension $K_i \subset K_{i+1}$ is either algebraic, or is obtained by adjoining to K_i a special transcendental element over K_i.

We announce the following theorem.

Theorem 2.1 (on strong non-solvability of linear differential equations). *A homogeneous linear differential equation over* K *can be solved by generalized admissible extensions if and only if it can be solved by generalized quadratures.*

In other words, if such an equation cannot be solved by generalized quadratures, then no special transcendental extensions can help solve it either.

In the present paper, we prove a weaker version of Theorem 2.1 which provides a criterion of solvability of homogeneous linear differential equations over K by admissible extension. It implies the following theorem.

Theorem 2.2. *If a homogeneous linear differential equation can be solved by admissible extensions, then it can be solved by a nested chain of extensions by integrals and by transcendental exponentials of integrals.*

§ 3. An illustrative example

In this section, we present a simple example which illustrates our approach and Liouville's method.

Let K be a differential field of meromorphic functions on some connected domain on a complex line which is closed under differentiation.

Consider the second order homogeneous equation

$$y'' + ay' + by = 0 \tag{3}$$

over K, i.e., $a = a(x)$ and $b = b(x)$ are functions from K. It is well known that *y is a non-zero solution of (3) if and only if its logarithmic derivative $u = y'/y$ satisfies the Riccati equation*

$$u' + au + b + u^2 = 0. \tag{4}$$

Thus, problems of solving equations (3) and (4) in finite terms are closely related, and for solvability by quadratures are in fact equivalent. If one wants to show that equation (3) is not solvable by quadratures, one can try to show it instead for equation (4). Let us demonstrate how one can make one step in proving non-solvability by quadratures of equation (4).

Assume that there are no solutions of the Riccati equation (4) in the differential field K. Let us show that an addition to the field K, say, an exponent y cannot help in solving (4). Our goal is to prove the following theorem.

THEOREM 3.1. *If an exponent y (i.e., a solution of the equation $y' = y$) is transcendental over K, and the Riccati equation (4) has a solution in the extension $K\langle y\rangle$ of the differential field K by y, then it has a solution in K.*

REMARK 3.1. Liouville's theory of integrability in finite terms has the following slogan:

If a "simple" equation has a solution given by an "allowed" formula, then it has to have (another) solution given by a "simple" allowed formula.

Theorem 3.1 demonstrates this slogan: here a simple equation is the equation (4), an allowed formula represents a solution as an element of $K\langle y\rangle$ and a simple allowed formula represents a solution as an element of K.

Let us prove the first two lemmas.

LEMMA 3.1 (Liouville's principle for a transcendental exponent). *An element $a \in K\langle y\rangle$ is representable in a unique way as a rational function in y over K. The derivative R' of $R = R(y)$ is given by*

$$R' = R'_x + R'_y y, \tag{5}$$

where $R'_x = \partial R/\partial x$, $R'_y = \partial R/\partial y$ and $\partial/\partial x$, $\partial/\partial y$ are the following differentiations of the field $K(y)$:

1) *$\partial/\partial x$ sends y to zero and coincides with the differentiation in K on coefficients*;
2) *$\partial/\partial y$ sends y to 1 and sends to zero all elements of the differential field K.*

Proof. Indeed, since $y' = y$, any element of the differential field $K\langle y\rangle$ is representable as a rational function in y over K. If an element a can be represented as a rational function in y in two different ways, i.e., $a = R_1(y)$; $a = R_2(y)$, then $R_1(y) = R_2(y)$, which means that y is algebraic over K and contradicts the assumption. One can check (5) using the chain rule and the identity $y' = y$. Lemma 3.1 is proved.

Remark 3.2. If y is algebraic element over K, then, for different rational functions R_1, R_2, the elements $R_1(y)$ and $R_2(y)$ could be equal. For some rational functions R the element $R(y)$ is not defined. Assume that an element y (either transcendental or algebraic over K) satisfies the equation $y' = y$ and for some R the element $a = R(y)$ is defined, then $a' = R'_x(y) + yR'_y(y)$ for its derivative a'.

According to Lemma 3.1, the field $K\langle y\rangle$ is isomorphic to the field $K(y)$ of rational functions over K with the differentiation given by (5). In the field of rational functions $K(y)$, the order of a rational function at the point $y = 0$ is defined.

Definition 3.1. A rational function $R(y)$ over K has order m at the point $y = 0$ if it is representable as $R = y^m R_0$, where R_0 is a rational function which is not divisible by any positive or negative degree of y.

Using (5), one can check the following lemma.

Lemma 3.2. *If a rational function R has order m at the point $y = 0$, then its derivative $R' = R'_x + R'_y y$ has order at $y = 0$ not smaller than m.*

Proof. If $R = y^m R_0$, then

$$R' = my^m R_0 + y^{m+1}(R_0)'_y + y^m(R_0)'_x.$$

The function R_0 equals to P/Q, where Q is not divisible by y. Thus, $(R_0)'_y$ and $(R_0)'_x$ are equal to $(P'_yQ - PQ'_y)/Q^2$ and $(P'_xQ - PQ'_x)/Q^2$, where the polynomial Q^2 is not divisible by y. So $(R_0)'_y$ and $(R_0)'_x$ have non-negative order, which implies Lemma 3.2.

Proof of Theorem 3.1. Assume that $u = R(y)$ satisfies (4). If we plug in to the rational function R any solution y of the equation $y' = y$, then we still will get a solution of (4) under the assumption that the plugging in of y to R is well defined (see Remark 3.2).

Let us try to plug in to R the solution $y \equiv 0$. If we succeed, then we will have a solution $R(0)$ of the Riccati equation (4) in the field K. One can do it only if the order of the rational function $R(y)$ at the point $y = 0$ is non-negative. If R has a pole at the point $y = 0$, then plugging $y = 0$ in to R makes no sense.

Let us show that if $u = R(y)$ has a negative order k at $y = 0$, then it cannot satisfy equation (4). Indeed, if u has an order $k < 0$, then u^2 has an order $2k < 0$. The terms u and u' in (4) have orders $\geqslant k$. So the term u^2 cannot be cancelled and Riccati's equation cannot be satisfied. Thus, if $u = R$ satisfies (4), it cannot have negative order at $y = 0$.

The theorem is proved.

§ 4. Generalized Riccati equation

In this section, we recall classical results on the generalized Riccati equation (the presentation is borrowed from [2]). Let y be a non-zero element of a differential field and let u be its logarithmic derivative, i.e., $y' = uy$.

Definition 4.1. Let D_n be a polynomial in u and in its derivatives $u', \dots, u^{(n-1)}$ up to order $(n-1)$ defined inductively by the following conditions:

$$D_0 = 1, \qquad D_{k+1} = \frac{dD_k}{dx} + uD_k.$$

Lemma 4.1. 1. *The polynomial D_n has integral coefficients and* $\deg D_n = n$. *The degree n homogeneous part of D_n equals to u^n (i.e., $D_n = u^n + \widetilde{D}_n$, where* $\deg \widetilde{D}_n < n$).

2. *If y is a function whose logarithmic derivative is equal to u (i.e., if $y' = uy$), then $y^{(n)} = D_n(u)y$ for any $n \geqslant 0$.*

Both claims of the lemma can be easily checked by induction.

Consider a homogeneous linear differential equation whose coefficients a_i belong to a differential field K:

$$y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = 0. \tag{6}$$

Definition 4.2. The equation

$$D_n + a_1 D_{n-1} + \dots + a_n D_0 = 0 \tag{7}$$

of order $(n-1)$ is called the *generalized Riccati equation* for the homogeneous linear differential equation (6).

Lemma 4.2. *A non-zero element y satisfies the linear differential equation (6) if and only if its logarithmic derivative $u = y'/y$ satisfies the corresponding generalized Riccati equation (7).*

Proof. For proving Lemma 4.2 in one direction, one can divide (6) by y and use the identity $y^{(k)}/y = D_k(u)$.

In the other direction, if u is a solution of (7), then multiply (7) by y and use the identity $y^k = D_k(u)y$. We obtain that y is a non-zero solution of (6). Lemma 4.2 is proved.

§ 5. Reduction of order

In this section, we recall the classical procedure of order reduction for homogeneous linear differential equations (the presentation is borrowed from [5]).

5.1. Division with a remainder of differential operators. A linear differential operator of order n over a differential field K is an operator $L = a_n D^n + \dots + a_0$, where $a_i \in K$ and $a_n \neq 0$, acting on an element y of any differential field F containing k by the formula

$$L(y) = a_n y^{(n)} + \cdots + a_0 y.$$

For operators L_1 and L_2 over K, their product $L = L_1 \circ L_2 = L_1(L_2)$ is also an operator over K. The multiplication of operators is, in general, non-commutative, but the leading term of the operator $L = L_1 \circ L_2$ is equal to the product of the leading terms of the operators L_1 and L_2. For operators L and L_2 of orders n and k over K, there exist unique operators L_1 and R over K such that $L = L_1 \circ L_2 + R$, and the order of R is strictly less than k. The operator R is called the *remainder in the right division* of the operator L by the operator L_2.

The operators L_1 and R can be constructed explicitly: the algorithm for the division of operators with a remainder is based on the formula for the leading term of the product and is very similar to the algorithm of the division with a remainder for polynomials in one variable.

5.2. Procedure of order reduction. Let L be a linear differential operator over K, y_1 a non-zero element of the field K, $p = y_1'/y_1$ its logarithmic derivative and $L_2 = D - p$ the first order operator annihilating y_1.

LEMMA 5.1. *The operator L is divisible by L_2 from the right if and only if the element y_1 satisfies the identity $L(y_1) \equiv 0$.*

PROOF. The remainder R in the right division of L by L_2 is the operator of multiplication by c_0, where $c_0 = (1/y_1)L(y_1)$. Indeed, the desired equality can be obtained by plugging in $y = y_1$ to the identity $L(y) \equiv L_1 \circ L_2(y) + c_0 y$. Lemma 5.1 is proved.

Using a non-zero solution y_1 of the equation $L(y) = 0$ of order n, one can reduce the order of this equation. To this end, one needs to represent the operator L in the form $L = L_1 \circ L_2$, where L_1 is an operator of order $(n - 1)$.

The coefficients of the operator L_1 lie in an extension of the differential field K obtained by adjoining the logarithmic derivative p of the element y_1. If one knows some solution u of the equation $L_1(u) = 0$, then one can construct a certain solution y of the initial equation $L(y) = 0$. To do that, it suffices to solve the equation $L_2(y) = y' - py = u$.

LEMMA 5.2. *An element y satisfies the equation $L_2(y) = u$ if and only if $y = y_1 z$, where z satisfies the equation $z' = u/y_1$.*

PROOF. The operator $L_2(y) = y' - (y_1'/y_1)y$ can be rewritten in the form $L_2(y) = y_1(y/y_1)'$. Lemma 5.2 is proved.

The procedure just described is called the *reduction of order* of a differential equation.

REMARK 5.1. An operator annihilating y_1 is defined up to a factor, which can be an arbitrary element of the field K, and the procedure of the order reduction depends on the choice of this element. It is easier to divide L from the right by the operator $\widetilde{L}_2 = D \circ y_1^{-1}$ and represent L in the form $L = \widetilde{L}_1 \widetilde{L}_2$ which reduces the initial equation $L(y) = 0$ to the equation $\widetilde{L}_1(u) = 0$ of smaller order. Usually, this very procedure of order reduction is given in textbooks on differential equations. Note that the coefficients of the operator $\widetilde{L}_1$ lie in the extension of the differential field K obtained by adjoining the element y_1 itself, rather than its logarithmic derivative p. This makes the operator $\widetilde{L}_1$ inconvenient for the purposes of our paper.

§ 6. Pure transcendental extensions

A field extension $K \subset F$ is *pure transcendental* if any element $y \in F$, which is algebraic over K, belongs to K. Let us start with the following example. Let y be an integral over K, i.e., $y' = a$ for some $a \in K$.

LEMMA 6.1. *The integral y either belongs to the field K or is transcendental over K.*

PROOF. Assume that an integral y of an element $a \in K$ is algebraic over K but does not belong to K. Let Q be a polynomial over K of the smallest possible degree $n > 1$ such that $Q(y) = a_n y^n + \cdots + a_0 = 0$. We can assume that $a_n = 1$. Differentiating the identity $Q(y) = 0$, we obtain the equation

$$(na + a'_{n-1})y^{n-1} + \cdots + (a_1 a + a'_0) = 0. \tag{8}$$

The coefficient $na + a'_{n-1}$ is not equal to zero. Indeed, otherwise the derivative of an element $-a_{n-1}/n$ is a. It means that y belongs to K, since $y = -a_{n-1}/n + C$ for some constant C, and all constants belong to K. Equation (8) for y has degree $n - 1 < n$. The contradiction proves that y is not algebraic over K. Lemma 6.1 is proved.

Lemma 6.1 implies that any non-trivial extension by adjoining an integral is pure transcendental. Let us describe all pure transcendental extensions of transcendental degree one.

Let $K \subset F$ be a nested pair of differential fields. Assume that $y \in F$ is transcendental over K and the algebraic field $K(y) \subset F$ generated by y and K is closed under differentiation, in particular, y' belongs to the algebraic field $K(y)$, i.e., $y' = R(y)$, where R is some rational function over K.

Liouville's principle shows that, for any $R \in K(y)$, there is a pure transcendental extension $K \subset K\langle y\rangle$ in which the identity $y' = R(y)$ holds and the rational function R together with differentiation in the field K completely determine the differential field $K\langle y\rangle$.

LEMMA 6.2. *Let K be a differential field, and $K(t) \supset K$ the field of rational functions over K in indeterminate t. Then, for any choice of rational function $\mathcal{R} \in K(t)$, there exists a unique differentiation on the field $K(t) \supset K$ which coincides with the differentiation on the subfield K and is equal to $\mathcal{R}(t)$ on t.*

PROOF. Consider the map sending $R \in K(t)$ to $R' \in K(t)$ given by the formula

$$R' = R'_K + \mathcal{R} R_t, \tag{9}$$

where R'_K denotes the derivative of R by the differentiation rule of K, viewing t as a constant. And R_t denotes the usual partial derivative with respect to t, viewing the elements of K as constant. One can check that the map $R \to R'$ provides a differentiation of the field $K(t)$ which satisfies all needed conditions. Such differentiation is unique, since $K(t)$ is generated by t and K. Lemma 6.2 is proved.

Lemma 6.3 (Liouville's Principle). *Let F be some differential field extension of K, and $y \in F$ transcendental over K. Suppose that y satisfies an equation $y' = \mathcal{R}(y)$, where $\mathcal{R}$ is a rational function with coefficients in K. Then $K\langle y\rangle$ is isomorphic to the field $K(t)$ of rational functions over K with differentiation given $t' = \mathcal{R}(t)$ (see Lemma 6.2).*

Proof. Consider the map from the field $K(t)$ of rational functions over K to the field $K\langle y\rangle$ which fixes elements of K and sends t to y. Since y is transcendental over K, this map is injective. Indeed, if $R_1(y) = R_2(y)$ for different functions $R_1, R_2 \in K(t)$, then the identity $R_1(y) = R_2(y)$ provides an algebraic equation over K on y which is impossible.

So, the field $K\langle y\rangle$ contains an isomorphic copy of (the algebraic field) $K(t)$, and by Lemma 6.2, $K(t)$ can be made a differential field with the choice $t' = \mathcal{R}(t)$. Our map preserves differentiation between the differential field $K(t)$ and its image in $K\langle y\rangle$. And since $K\langle y\rangle$ is the smallest differential field containing K and y, we must have that $K\langle y\rangle$ is precisely the image of $K(t)$. Lemma 6.3 is proved.

The following lemma is obvious.

Lemma 6.4. *Let $K = K_0 \subset \cdots \subset K_n = F$ be a chain of fields. If, for any $i = 0, \ldots, n-1$, the extension $K_i \subset K_{i+1}$ is pure transcendental, then so is the extension $K \subset F$.*

Proof. Assume that $y \in K_{i+1}$ is algebraic over K_0. Since $K_0 \subset K_i$, the element y belongs to K_i. By repeating this argument, one obtains that $y \in K_0$. Lemma 6.4 is proved.

Consider a homogeneous linear differential equation (6) over the field $\mathbb{C}(z)$ of complex rational functions with the usual differentiation.

Lemma 6.5. *If all coefficients of equation (6) are polynomials $a_i \in \mathbb{C}[z]$, then the extension M of the field $\mathbb{C}(z)$, obtained by adjoining all solutions of (6), is pure transcendental.*

Proof. The extension M is contained in the field of meromorphic functions on the complex line. A meromorphic function is algebraic over $\mathbb{C}(z)$ if and only if it is rational. Lemma 6.5 is proved.

§ 7. Adjoining a special transcendental element

In this section, we prove Lemma 7.2 which provides an important property of special transcendental extensions.

For a rational function $R \in K(y)$ over a field K, we denote by $\operatorname{ord} R$ its order at the point $y=0$ (see Definition 3.1). For any functions $R_1, R_2 \in K(y)$, the following relations hold:

1) $\operatorname{ord}(R_1R_2) = \operatorname{ord} R_1 + \operatorname{ord} R_2$;

2) $\operatorname{ord}(R_1 + R_2) \geqslant \min(\operatorname{ord} R_1, \operatorname{ord} R_2)$, and if $\operatorname{ord} R_1 \neq \operatorname{ord} R_2$, then $\operatorname{ord}(R_1 + R_2) = \min(\operatorname{ord} R_1, \operatorname{ord} R_2)$.

Let $D\colon K(y) \to K(y)$ be any differentiation of the field $K(y)$.

Lemma 7.1. *Non-zero elements $f \in K(y)$ satisfying inequality $\operatorname{ord} D(f) \geqslant \operatorname{ord} f$ form a multiplicative subgroup in the field $K(y)$.*

Proof. The inequality $\operatorname{ord} D(f) \geqslant \operatorname{ord} f$ can be rewritten as $\operatorname{ord}(Df/f) \geqslant 0$. The logarithmic derivative Df/f provides a homomorphism of the multiplicative group of the field $K(y)$ to its additive group. An element f satisfies $\operatorname{ord} Df \geqslant \operatorname{ord} f$ if and only if it belongs to the pre-image of the additive subgroup of functions $g \in K(y)$ having non-negative order. Lemma 7.1 is proved.

Let y be a special transcendental element over a differential field K satisfying the differential equation $y' = \mathcal{R}(y) = yR_0(y)$, where R_0 has non-negative order at $y = 0$, i.e., $\operatorname{ord} R_0 \geqslant 0$. The differential field $K\langle y\rangle$ is isomorphic to the field $K(y)$ with the differentiation $R' = R'_K + yR_0R_y$.

Lemma 7.2. *For any element R in the differential field $K\langle y\rangle$,*

$$\operatorname{ord} R' \geqslant \operatorname{ord} R. \tag{10}$$

Proof. Indeed, $R' = R'_K + yR_0R'_y$. It is easy to check that $\operatorname{ord} R'_K \geqslant \operatorname{ord} R$ and $\operatorname{ord} yR'_y \geqslant \operatorname{ord} R$. Indeed, these inequalities obviously hold for polynomials over K. Now one can apply Lemma 7.1, and the lemma follows from the properties of order listed above. Lemma 7.2 is proved.

§ 8. Existence of a solution in an admissible extension

In this section, we prove the following theorem.

Theorem 8.1. *A homogeneous linear differential equation* (6) *has a solution in some admissible extension of the field K if and only if it has a solution y whose logarithmic derivative belongs to K, and y either belongs to K, or is transcendental over K.*

The proof is based on the following lemma.

Lemma 8.1. *Let $K \subset K\langle y\rangle$ be an extension by adjoining a special transcendental element y over K. Then the generalized Riccati equation* (7) *for equation* (6) *has a solution in $K\langle y\rangle$ if and only if it has a solution in the field K.*

Let T be a polynomial over K in u and in its derivatives $u, u', \dots, u^{(k)}, \dots$.

Definition 8.1. We say that T of some degree n is a *Rosenlicht type polynomial* if the degree n homogeneous part of T is equal to u^n (i.e., $T = u^n + \widetilde{T}$, where $\deg \widetilde{T} < n$).

An equation

$$T(u, u', \dots, u^{(n)}) = 0, \tag{11}$$

where T is a Rosenlicht type polynomial, is called a *Rosenlicht type equation.*

The generalized Riccati equation is a Rosenlicht type equation. This property of the generalized Riccati equation plays the key role in the classical approach (which goes back to Liouville and was specified by Rosenlicht) to the problem of solvability of homogeneous linear differential equations in finite terms. Thus, instead of Lemma 8.1, we prove the following more general lemma.

Lemma 8.2. *Let $K \subset K\langle y\rangle$ be an extension by adjoining a special transcendental element y over K. An equation (11) has a solution in $K\langle y\rangle$ if and only if it has a solution in the field K.*

Proof. Let $R(y) \in K\langle y\rangle$ where y is a special transcendental element over K. Let us show that if $R(y)$ satisfies (11), then $\operatorname{ord} R = m$ is non-negative. Indeed, if $m < 0$, then $\operatorname{ord} u^n = nm$ is strictly smaller than $\operatorname{ord} \widetilde{T}$, which follows from the definition of Rosenlicht polynomial and Lemma 7.2. Thus, $m \geqslant 0$. So, one can evaluate both sides of (11) at $y = 0$ and obtain a solution of (11) in the field K. Lemma 8.1 is proved.

Corollary 8.1. *An equation (11) has a solution in an admissible extension E of the field K if and only if it has a solution in the field K.*

Proof. Let $K_0 \subset \cdots \subset K_n = E$ be a chain in which each extension $K_i \subset K_{i+1}$ is obtained by adjoining a special transcendental element over K_i. By Lemma 8.2, if (11) has a solution in K_{i+1}, it also has a solution in K_i. This argument proves corollary. Corollary 8.1 is proved.

Proof of Theorem 8.1. By Lemma 4.2, the logarithmic derivative of any non-zero solution y of (6) has to satisfy its generalized Riccati equation. Assume that y belongs to an admissible extension E of K. Corollary 8.1 implies that the generalized Riccati equation has a solution $u \in K$. Thus, (6) has a non-zero solution z such that $z'/z = u \in K$. If this element z is algebraic over K, it belongs to K, since E is a pure transcendental extension of K. Theorem 8.1 is proved.

In § 9 and § 10, we study extensions of a differential field K by exponential of integrals over K. Related material can be found in [8].

§ 9. Algebraic relations between exponentials of integrals

Let K be a differential field. For y belonging to the multiplicative group $K^* = K \setminus \{0\}$ of the field K, the logarithmic derivative y'/y is defined. It is easy to check the following lemma.

LEMMA 9.1. *The map, which sends each element $y \in K^*$ to its logarithmic derivative, provides a homomorphism of the multiplicative group K^* to the additive group K_+ of the field K, i.e., for any $u, v \in K^*$ any integers k, n the following identity holds*:

$$\frac{(u^k v^n)'}{u^k v^n} = k\frac{u'}{u} + n\frac{v'}{v}.$$

DEFINITION 9.1. Let F be any extension of a differential field K. A non-zero element $y \in F$ is an *exponential of integral* of $a \in K$ if $y' = ay$. In other words, $y \in F^*$ is an exponential of integral of $a \in K$ if a is the logarithmic derivative of y.

Let F be an extension of K.

LEMMA 9.2. *Assume that a non-zero element $y \in F$ satisfies an equation $y' = ay$ for $a \in K$ and an element $u \in F$ is representable in a form $u = cy^m$, where $c \in K$, $m \in \mathbb{Z}$. Then*

1) *the derivative u' of u also is representable in the form $u' = \widetilde{c}y^m$, where $\widetilde{c} = c' + ma$;*

2) *$u' = 0$ if and only if $c = \lambda y^{-m}$, where λ is a constant, i.e., $\lambda' = 0$. If $u' = 0$, then $y^m \in K$.*

PROOF. The first relation follows from a direct differentiation. If $u' = 0$, then u is a constant λ. Thus, $cy^m = \lambda$, which implies that $c = \lambda y^{-m}$ and $y^m = c^{-1}\lambda$. Lemma 9.2 is proved.

DEFINITION 9.2. Let $F = K\langle y\rangle$ be an extension of a differential field K by an element y. Then F is an *extension by exponential of integral of* $a \in K$ (a *transcendental extension by exponential of integral of* $a \in K$) if $y' = ay$ (if y is transcendental over K and $y' = ay$).

THEOREM 9.1. *Let $y_1, \dots, y_n \in F$ be the exponentials of integrals of elements $a_1, \dots, a_n \in K$, where K is a differential subfield of F. Assume that there are no non-trivial monomials $u = y_1^{k_1} \cdots y_n^{k_n}$, where $k_1, \dots, k_n$ are not necessary positive integers* (*assuming that, at least, one $k_i \neq 0$*) *such that a relation $u = c$ for some $c \in K$ holds. Then the elements $y_1, \dots, y_n$ are algebraically independent over the field K.*

PROOF. Assume that $y_1, \dots, y_n$ are algebraically dependent and let

$$P = \sum c_{m_1,\dots,m_n} x_1^{m_1} \dots x_n^{m_n}$$

be a non-zero Laurent polynomial over K (i.e., $m_1, \dots, m_n$ are not necessarily positive integral numbers) which annihilates $y_1, \dots, y_n$, i.e., $\sum c_{m_1,\dots,m_n} y_1^{m_1} \cdots y_n^{m_n} = 0$. One can choose such a Laurent polynomial P which contains the smallest possible number of non-zero terms $c_{m_1,\dots,m_n} y_1^{m_1} \cdots y_n^{m_n}$. By dividing the relation $P = 0$ by one of its terms, one obtains a Laurent polynomial $\widetilde{P}$ having the same number of terms as P, but one of its terms is equal to 1. By Lemma 9.2, the derivative u' of the term $u = c_{m_1,\dots,m_n} y_1^{m_1} \cdots y_n^{m_n}$ has a form $u' = \widetilde{c}_{m_1,\dots,m_n} y_1^{m_1} \cdots y_n^{m_n}$ for some $\widetilde{c}_{m_1,\dots,m_n} \in K$.

Thus, by differentiating $\widetilde{P}=0$, one obtains a relation $\widetilde{P}'(y_1,\dots,y_n)=0$ containing smaller number of terms (the derivative of the term 1 is zero). Thus, the derivative of each term of the Laurent polynomial $\widetilde{P}$ is equal to zero. By Lemma 9.2, this means that some non-trivial monomial $y_1^{m_1}\cdots y_n^{m_n}$ belongs to the field K. Theorem 9.1 is proved.

§ 10. Transcendental extensions by exponentials of integrals

Assume that a differential field $F=K\langle y_1,\dots,y_n\rangle$ is generated over a field K by non-zero elements $y_1,\dots,y_n$, such that $y_i'=a_iy_i$, where a_i belong to K. Let G be the multiplicative subgroup in F^* generated by elements $y_1,\dots,y_n$, and let G_0 be its subgroup which is equal to the intersection of G with the field K, $G_0=G\cap K^*$. Denote by $\Psi_0\subset\Psi$ the images of the multiplicative groups G_0,G under the map which sends $y\in F^*$ to $y'/y\in F_+$. By construction, the group Ψ is generated by the elements $a_i\in K_+$.

Theorem 10.1. *The factor group G/G_0 has the following properties.*

1. *If G/G_0 is torsion free, then there is a chain of fields $K=K_0\subset\cdots\subset K_m=F$ such that, for $i=0,\dots,m-1$, the field K_{i+1} is equal to $K_i\langle u_{i+1}\rangle$, where u_{i+1} is an exponent of an integral over K_i, transcendental over K_i.*

2. *If G/G_0 has non-trivial torsion, then there is no chain of fields $K=K_0\subset\cdots\subset K_m\supset F$ such that, for $i=0,\dots,m-1$, the field K_{i+1} is a pure transcendental extension of the field K_i.*

Proof. 1. Let $u_1,\dots,u_m$ be generators of the factor group G/G_0. The elements $u_1,\dots,u_m$ can be chosen to be exponentials of integrals over the field K_0. Each monomial in $u_1,\dots,u_m$ does not belong to the subgroup $G_0=G\cap K_0$, since the factor group is torsion free. Thus, by Theorem 9.1, elements $u_1,\dots,u_m$ are algebraically independent over K_0. Let K_i for $i=1,\dots,m$ be the field $K_0\langle u_1,\dots,u_i\rangle$. By construction, in the chain of fields, $K_0\subset K_1\cdots\subset K_m$, for $i=0,\dots,m-1$, the field K_{i+1} is the extension of K_i by exponent of integral u_{i+1} over the field $K_0\subset K_i$. The field K_m is equal to the field F. All these extensions $K_i\subset K_{i+1}$ are pure transcendental, since $u_1,\dots,u_m$ are algebraically independent over K_0.

2. If the group G/G_0 has non-trivial torsion, then there is a monomial u in $y_1,\dots,y_n$ which does not belong to G_0 but some positive power of it does. Thus, u does not belong to the field K_0 but some positive power of it does.

The monomial u is an algebraic element over K_0 which does not belong to K_0. Any chain $K_0\subset\cdots\subset K_m$ of pure transcendental extensions provides a transcendental extension $K_0\subset K_m$ which cannot contain the element $u\in F$. Theorem 10.1 is proved.

§ 11. Criterion of solvability in admissible extensions

Let $L=a_nD^n+\cdots+a_0$, where $a_i\in K$ is a linear differential operator of order n over K. In this section, we will state and prove a criterion of solvability of the homogeneous equation

$$L(y) = a_n y^{(n)} + \cdots + a_1 y = 0 \tag{12}$$

over K by an admissible extension E of K.

THEOREM 11.1. *All solutions of* (12) *belong to an admissible extension* $E \supset K$ *if and only if the following two conditions simultaneously hold.*

1. *There is a sequence of non-zero elements* $z_1, \ldots, z_n$ *which are exponential of integrals over* K, *i.e.,* $z_1' = p_1 z_1, \ldots, z_n' = p_n z_n$ *and* $p_1, \ldots, p_n \in K$ *such that the operator* L *is representable in the form*

$$L = a_n L_n \circ \cdots \circ L_1, \tag{13}$$

where each operator L_i *is an operator of order one. Moreover,* L_i *is equal to* $D - p_i$.

2. *Let* G *be the multiplicative group generated by* $z_1, \ldots, z_n$ *and let* G_0 *be the intersection of* G *with* K, $G_0 = G \cap K$. *Then the factor group* G/G_0 *is torsion free.*

PROOF. If there is an admissible extension E containing all solutions of (13), then by Theorem 8.1, the equation has to have a solution z_1 whose logarithmic derivative p_1 belongs to K. Let L_1 be the operator $y' - p_1 y_1$. Let us divide L by L_1 from the right, $L = \widetilde{L}_1 \circ L_1$. The coefficients of $\widetilde{L}_1$ belongs to K, since $p_1 \in K$. Each solution u of $\widetilde{L}_1(y) = 0$ is representable in the form $u = L_1(y)$, where y is a solution of the original equation. So all solutions of $\widetilde{L}_1(y) = 0$ also belong to the admissible extension E of the field K.

Thus, one can apply Theorem 8.1 to the equation $\widetilde{L}_1(y) = 0$ to find its solution z_2 whose logarithmic derivative p_2 belongs to K. Let L_2 be the operator $D - p_2$ and let us represent $\widetilde{L}_1$ in the form $\widetilde{L}_1 = \widetilde{L}_2 \circ L_2$.

Repeating this process, one obtains a sequence of exponentials of integrals of $p_1, \ldots, p_n \in K$ for which decomposition (13) of the operator L holds.

The exponential of integrals $z_1, \ldots, z_n$ belong to an admissible extension E. By Theorem 10.1, the group G/G_0 is torsion free.

The criterion for solvability is proved in one direction. Let us prove it in the opposite direction. If the group G/G_0 is torsion free, by Theorem 10.1, one can construct an admissible extension E_1 of K which contains all the exponential of integrals $z_1, \ldots, z_n$. Over the field E_1 the operator L is decomposed as a product of factors $D - p_i$. Thus, the equation $L = 0$ can solved step by step, thereby solving the equations $L_i(y) = u$. Since $z_i \in E$, solutions of these equation can be obtained in extensions by integrals (see Lemma 5.2). By Lemma 6.1, such extensions are pure transcendental. So, extending E_1 by a chain of extensions obtained by adjoining integrals one at a time, one obtains an admissible extension E which contains all solutions of (12).

Note that if the group G/G_0 has non-trivial torsion, then the extension of K by adjoining all solutions of (12) is not pure transcendental. If it is pure transcendental (as in Lemma 6.5), then the group G/G_0 automatically has no torsion. Such an equation is solvable by admissible extension if and only if the operator L admits a decomposition (13). Theorem 11.1 is proved.

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Поступило в редакцию
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